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# Fixed point theory of cyclical generalized contractive conditions in partial metric spaces

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**Abstract**

The purpose of this paper is to study fixed point theorems for a mapping satisfying the cyclical generalized contractive conditions in complete partial metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.

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**Keywords:** fixed point; cyclic $\mathcal{CW}$-contraction; cyclic $\mathcal{MK}$-contraction; partial metric space

## 1 Introduction and preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let $(X,d)$ be a metric space, $D$ be a subset of $X$ and $f : D \to X$ be a map. We say $f$ is contractive if there exists $\alpha \in [0,1)$ such that for all $x, y \in D$,

$$d(fx, fy) \le \alpha \cdot d(x,y).$$

The well-known Banach fixed point theorem asserts that if $D = X$, $f$ is contractive and $(X,d)$ is complete, then $f$ has a unique fixed point in $X$. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, in 1969, Boyd and Wong [2] introduced the notion of $\Phi$-contraction. A mapping $f : X \to X$ on a metric space is called $\Phi$-contraction if there exists an upper semi-continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that

$$d(fx, fy) \le \Phi\bigl(d(x,y)\bigr) \quad \text{for all } x, y \in X.$$

In 1994, Mattews [3] introduced the following notion of partial metric spaces.

**Definition 1** [3] A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

($p_1$) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
($p_2$) $p(x,x) \le p(x,y)$;

($p_3$) $p(x,y) = p(y,x)$;
($p_4$) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair $(X,p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

**Remark 1** It is clear that if $p(x,y) = 0$, then from ($p_1$) and ($p_2$), $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric $p$ on $X$ generates a $\mathcal{T}_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\gamma) : x \in X, \gamma > 0\}$, where $B_p(x,\gamma) = \{y \in X : p(x,y) < p(x,x) + \gamma\}$ for all $x \in X$ and $\gamma > 0$. If $p$ is a partial metric on $X$, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on $X$.

We recall some definitions of a partial metric space as follows.

**Definition 2** [3] Let $(X,p)$ be a partial metric space. Then
(1) a sequence $\{x_n\}$ in a partial metric space $(X,p)$ converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$;
(2) a sequence $\{x_n\}$ in a partial metric space $(X,p)$ is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists (and is finite);
(3) a partial metric space $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x,x) = \lim_{m,n\to\infty} p(x_m,x_n)$;
(4) a subset $A$ of a partial metric space $(X,p)$ is closed if whenever $\{x_n\}$ is a sequence in $A$ such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

**Remark 2** The limit in a partial metric space is not unique.

**Lemma 1** [3, 4]
(a) $\{x_n\}$ *is a Cauchy sequence in a partial metric space* $(X,p)$ *if and only if it is a Cauchy sequence in the metric space* $(x,d_p)$;
(b) *a partial metric space* $(X,p)$ *is complete if and only if the metric space* $(X,d_p)$ *is complete. Furthermore,* $\lim_{n\to\infty} d_p(x_n,x) = 0$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_m)$.

In 2003, Kirk, Srinivasan and Veeramani [5] introduced the following notion of the cyclic representation.

**Definition 3** [5] Let $X$ be a nonempty set, $m \in \mathbb{N}$ and $f : X \to X$ be an operator. Then $X = \bigcup_{i=1}^m A_i$ is called a cyclic representation of $X$ with respect to $f$ if
(1) $A_i$, $i = 1,2,\ldots,m$ are nonempty subsets of $X$;
(2) $f(A_1) \subset A_2, f(A_2) \subset A_3, \ldots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Kirk, Srinivasan and Veeramani [5] also proved the following theorem.

**Theorem 1** [5] *Let $(X,d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$, be closed nonempty subsets of $X$ and $X = \bigcup_{i=1}^m A_i$. Suppose that $f$ satisfies the following condition*:

$$d(fx,fy) \le \psi\bigl(d(x,y)\bigr), \quad \textit{for all } x \in A_i, y \in A_{i+1}, i \in \{1,2,\ldots,m\},$$

*where $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous from the right and $0 \le \psi(t) < t$ for $t > 0$. Then $f$ has a fixed point $z \in \bigcap_{i=1}^n A_i$.*

Recently, the fixed theorems for an operator $f : X \to X$ defined on a metric space $X$ with a cyclic representation of $X$ with respect to $f$ have appeared in the literature (see, *e.g.*, [6–8]). In 2010, Păcurar and Rus [7] introduced the following notion of a cyclic weaker $\varphi$-contraction.

**Definition 4** [7] Let $(X,d)$ be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be closed nonempty subsets of $X$ and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \to X$ is called a cyclic weaker $\varphi$-contraction if

(1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of $X$ with respect to $f$;
(2) there exists a continuous, non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for $t \in (0,\infty)$ and $\varphi(0) = 0$ such that

$$d(fx,fy) \le d(x,y) - \varphi\bigl(d(x,y)\bigr)$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, where $A_{m+1} = A_1$.

And Păcurar and Rus [7] proved the following main theorem.

**Theorem 2** [7] *Let $(X,d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be closed nonempty subsets of $X$ and $X = \bigcup_{i=1}^m A_i$. Suppose that $f$ is a cyclic weaker $\varphi$-contraction. Then $f$ has a fixed point $z \in \bigcap_{i=1}^n A_i$.*

In the recent years, fixed point theory has developed rapidly on cyclic contraction mappings, see [9–15].

The purpose of this paper is to study fixed point theorems for a mapping satisfying the cyclical generalized contractive conditions in complete partial metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.

## 2 Fixed point theorems (I)

In the section, we denote by $\Psi$ the class of functions $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying the following conditions:

($\psi_1$) $\psi$ is an increasing and continuous function in each coordinate;
($\psi_2$) for $t \in \mathbb{R}^+$, $\psi(t,t,t) \le t$, $\psi(t,0,0) \le t$ and $\psi(0,0,t) \le t$.

Next, we denote by $\Theta$ the class of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

($\varphi_1$) $\varphi$ is continuous and non-decreasing;
($\varphi_2$) for $t > 0$, $\varphi(t) > 0$ and $\varphi(0) = 0$.

And we denote by $\Phi$ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

($\phi_1$) $\phi$ is continuous;
($\phi_2$) for $t > 0$, $\phi(t) > 0$ and $\phi(0) = 0$.

We now state a new notion of cyclic $\mathcal{CW}$-contractions in partial metric spaces as follows.

**Definition 5** Let $(X, p)$ be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \to Y$ is called a cyclic $\mathcal{CW}$-contraction if

(1) $\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;
(2) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$\varphi\bigl(p(fx, fy)\bigr) \le \psi\bigl(\varphi\bigl(p(x, y)\bigr), \varphi\bigl(p(x, fx)\bigr), \varphi\bigl(p(y, fy)\bigr)\bigr) - \phi\bigl(M(x, y)\bigr), \tag{2.1}$$

where $\psi \in \Psi$, $\varphi \in \Theta$, $\phi \in \Phi$, and $M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy)\}$.

**Theorem 3** *Let $(X, p)$ be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y \to Y$ be a cyclic $\mathcal{CW}$-contraction. Then $f$ has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.*

*Proof* Given $x_0$ and let $x_{n+1} = fx_n = f^n x_0$ for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \ne x_n$ for any $n = 0, 1, 2, \ldots$. Notice that for any $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$.

Step 1. We will prove that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \quad \text{that is,} \quad \lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0.$$

Using (2.1), we have

$$\begin{aligned}
\varphi\bigl(p(x_n, x_{n+1})\bigr) &= \varphi\bigl(p(fx_{n-1}, fx_n)\bigr) \\
&\le \psi\bigl(\varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_{n-1}, fx_{n-1})\bigr), \varphi\bigl(p(x_n, fx_n)\bigr)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr) \\
&= \psi\bigl(\varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr),
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\bigl\{p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n)\bigr\} \\
&= \max\bigl\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\bigr\}.
\end{aligned}$$

If $M(x_{n-1}, x_n) = p(x_n, x_{n+1})$, then

$$\begin{aligned}
\varphi\bigl(p(x_n, x_{n+1})\bigr) &\le \psi\bigl(\varphi\bigl(p(x_n, x_{n+1})\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr)\bigr) - \phi\bigl(p(x_n, x_{n+1})\bigr) \\
&\le \varphi\bigl(p(x_n, x_{n+1})\bigr) - \phi\bigl(p(x_n, x_{n+1})\bigr),
\end{aligned}$$

which implies that $\phi(p(x_n, x_{n+1})) = 0$, and hence $p(x_n, x_{n+1}) = 0$. This contradicts our initial assumption.

From the above argument, we have that for each $n \in \mathbb{N}$,

$$\varphi\bigl(p(x_n, x_{n+1})\bigr) \le \varphi\bigl(p(x_{n-1}, x_n)\bigr) - \phi\bigl(p(x_{n-1}, x_n)\bigr), \tag{2.2}$$

and

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

And since the sequence $\{p(x_n, x_{n+1})\}$ is decreasing, it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (2.2) and by the continuity of $\varphi$ and $\phi$, we get

$$\varphi(\eta) \le \varphi(\eta) - \phi(\eta),$$

and so we conclude that $\phi(\eta) = 0$ and $\eta = 0$. Thus, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.3}$$

By $(p_2)$, we also have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.4}$$

Since $d_p(x, y) \le 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, using (2.3) and (2.4), we obtain that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.5}$$

Step 2. We show that $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$. We claim that the following result holds.

**Claim** For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \ge n$ with $r - q = 1 \bmod m$, then $d_p(x_r, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $r_n, q_n \in \mathbb{N}$ with $r_n > q_n \ge n$ with $r_n - q_n = 1 \bmod m$ satisfying

$$d_p(x_{q_n}, x_{r_n}) \ge \epsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \ge n$ use, we can choose $r_n$ in such a way it is the smallest integer with $r_n > q_n \ge n$ satisfying $r_n - q_n = 1 \bmod m$ and $d_p(x_{q_n}, x_{r_n}) \ge \epsilon$. Therefore, $d_p(x_{q_n}, x_{r_n - m}) \le \epsilon$ and

$$\begin{aligned}
\epsilon &\le d_p(x_{q_n}, x_{r_n}) \\
&\le d_p(x_{q_n}, x_{r_n-m}) + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}) \\
&< \epsilon + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}).
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_n}, x_{r_n}) = \epsilon. \tag{2.6}$$

On the other hand, we can conclude that

$$\begin{aligned}
\epsilon &\le d_p(x_{q_n}, x_{r_n}) \\
&\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}) \\
&\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{q_n}) + d_p(x_{q_n}, x_{r_n}) + d_p(x_{r_n}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}).
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{2.7}$$

Since $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and using (2.4), (2.6) and (2.7), we have that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \frac{\epsilon}{2}, \tag{2.8}$$

and

$$\lim_{n\to\infty} p(x_{q_{n+1}}, x_{r_{n+1}}) = \frac{\epsilon}{2}. \tag{2.9}$$

Since $x_{q_n}$ and $x_{r_n}$ lie in different adjacently labeled sets $A_i$ and $A_{i+1}$ for certain $1 \le i \le m$, by using the fact that $f$ is a cyclic $\mathcal{CW}$-contraction, we have

$$\begin{aligned}
\varphi\big(p(fx_{q_n+1}, fx_{r_n+1})\big) &= \varphi\big(p(fx_{q_n}, fx_{r_n})\big) \\
&\le \psi\big(\varphi\big(p(x_{q_n}, x_{r_n})\big), \varphi\big(p(x_{q_n}, fx_{q_n})\big), \varphi\big(p(x_{r_n}, fx_{r_n})\big)\big) \\
&\quad - \phi\big(M(x_{q_n}, x_{r_n})\big) \\
&= \psi\big(\varphi\big(p(x_{q_n}, x_{r_n})\big), \varphi\big(p(x_{q_n}, x_{q_n+1})\big), \varphi\big(p(x_{r_n}, x_{r_n+1})\big)\big) \\
&\quad - \phi\big(M(x_{q_n}, x_{r_n})\big),
\end{aligned}$$

where

$$M(x_{q_n}, x_{r_n}) = \max\big\{p(x_{q_n}, x_{r_n}), p(x_{q_n}, x_{q_n+1}), p(x_{r_n}, x_{r_n+1})\big\}.$$

Thus, letting $n \to \infty$, we can conclude that

$$\varphi\left(\frac{\epsilon}{2}\right) \le \psi\left(\varphi\left(\frac{\epsilon}{2}\right), \varphi(0), \varphi(0)\right) - \phi\left(\frac{\epsilon}{2}\right) \le \varphi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}\right),$$

which implies $\phi(\frac{\epsilon}{2}) = 0$, that is, $\epsilon = 0$. So, we get a contradiction. Therefore, our claim is proved.

In the sequel, we will show that $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$. Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $r, q \geq n_1$ with $r - q = 1 \bmod m$, then

$$d_p(x_r, x_q) \leq \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d_p(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m}$$

for any $n \geq n_2$.

Let $r, q \geq \max\{n_1, n_2\}$ and $r > q$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $r - q = k \bmod m$. Therefore, $r - q + j = 1 \bmod m$ for $j = m - k + 1$, and so we have

$$\begin{aligned} d_p(x_q, x_r) &\leq d_p(x_q, x_{r+j}) + d_p(x_{r+j}, x_{r+j-1}) + \cdots + d_p(x_{r-1}, x_r) \\ &\leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \\ &\leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$.

Step 3. We show that $f$ has a fixed point $\nu$ in $\bigcap_{i=1}^m A_i$.

Since $Y$ is closed, the subspace $(Y, p)$ is complete. Then from Lemma 1, we have that $(Y, d_p)$ is complete. Thus, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, \nu) = 0.$$

And it follows from Lemma 1 that we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.10}$$

On the other hand, since the sequence $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$, we also have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0.$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, we can deduce that

$$\lim_{n\to\infty} p(x_n, x_m) = 0. \tag{2.11}$$

Since $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of $X$ with respect to $f$, the sequence $\{x_n\}$ has infinite terms in each $A_i$ for $i \in \{1, 2, \dots, m\}$. Now, for all $i = 1, 2, \dots, m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to $\nu$. Using (2.10) and (2.11), we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n\to\infty} p(x_{n_k}, \nu) = 0.$$

By (2.1),

$$\begin{aligned}\varphi\big(p(x_{n_{k+1}},f\nu)\big) &= \varphi\big(p(fx_{n_k},f\nu)\big)\\ &\le \psi\big(\varphi\big(p(x_{n_k},\nu)\big),\varphi\big(p(x_{n_k},fx_{n_k})\big),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(M(x_{n_k},\nu)\big)\\ &= \psi\big(\varphi\big(p(x_{n_k},\nu)\big),\varphi\big(p(x_{n_k},x_{n_k+1})\big),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(M(x_{n_k},\nu)\big),\end{aligned}$$

where

$$M(x_{n_k},\nu) = \max\big\{p(x_{n_k},\nu),p(x_{n_k},x_{n_k+1}),p(\nu,f\nu)\big\}.$$

Letting $k\to\infty$, we have

$$\begin{aligned}\varphi\big(p(\nu,f\nu)\big) &\le \psi\big(\varphi(0),\varphi(0),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(p(\nu,f\nu)\big)\\ &\le \varphi\big(p(\nu,f\nu)\big) - \phi\big(p(\nu,f\nu)\big),\end{aligned}$$

which implies $\phi(p(\nu,f\nu)) = 0$, that is, $p(\nu,f\nu) = 0$. So, $\nu = f\nu$.

Step 4. Finally, to prove the uniqueness of the fixed point, suppose that $\mu$, $\nu$ are fixed points of $f$. Then using the inequality (2.1), we obtain that

$$\begin{aligned}\varphi\big(p(\mu,\nu)\big) = \varphi\big(p(f\mu,f\nu)\big) &\le \psi\big(\varphi\big(p(\mu,\nu)\big),\varphi\big(p(\mu,f\mu)\big),\varphi\big(p(\nu,f\nu)\big)\big)\\ &\quad - \phi\big(M(\mu,\nu)\big),\end{aligned}$$

where

$$M(\mu,\nu) = \max\big\{p(\mu,\nu),p(\mu,f\mu),p(\nu,f\nu)\big\} = p(\mu,\nu).$$

So, we also deduce that

$$\begin{aligned}\varphi\big(p(\mu,\nu)\big) &\le \psi\big(\varphi\big(p(\mu,\nu)\big),0,0\big)\\ &\le \varphi\big(p(\mu,\nu)\big) - \phi\big(p(\mu,\nu)\big),\end{aligned}$$

which implies that $\phi(p(\mu,\nu)) = 0$, and hence $p(\mu,\nu) = 0$, that is, $\mu = \nu$. So, we complete the proof. □

The following provides an example for Theorem 3.

**Example 1** Let $X = [0,1]$ and $A = [0,1]$, $B = [0,\frac{1}{2}]$, $C = [0,\frac{1}{4}]$. We define the partial metric $p$ on $X$ by

$$p(x,y) = \max\{x,y\} \quad \text{for all } x,y\in X,$$

and define the function $f : X \to X$ by

$$f(x) = \frac{x^2}{1+x} \quad \text{for all } x\in X.$$

Now, we let $\varphi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ be

$$\varphi(t) = 2t, \qquad \phi(t) = \frac{2t}{5(1+t)} \quad \text{and} \quad \psi(t) = \frac{4}{5} \cdot \max\{t_1, t_2, t_3\}.$$

Then $f$ is a cyclic $\mathcal{CW}$-contraction and 0 is the unique fixed point.

*Proof* We claim that $f$ is a cyclic $\mathcal{CW}$-contraction.

(1) Note that $f(A) = [0, \frac{1}{2}] \subset B$, $f(B) = [0, \frac{1}{6}] \subset C$ and $f(C) = [0, \frac{1}{20}] \subset A$. Thus, $A \cup B \cup C$ is a cyclic representation of $X$ with respect to $f$;

(2) For $x \in A$ and $y \in B$ (or, $x \in B$ and $y \in C$), without loss of generality, we may assume that $x \geq y$, then we have

$$\begin{aligned}
&\varphi\big(p(fx, fy)\big) = \varphi\left(p\left(\frac{x^2}{1+x}, \frac{y^2}{1+y}\right)\right) = \varphi\left(\frac{x^2}{1+x}\right) = \frac{2x^2}{1+x}, \\
&\psi\big(\varphi\big(p(x,y)\big), \varphi\big(p(x,fx)\big), \varphi\big(p(y,fy)\big)\big) \\
&\quad = \psi\left(\varphi\big(p(x,y)\big), \varphi\left(p\left(x, \frac{x^2}{1+x}\right)\right), \varphi\left(p\left(y, \frac{y^2}{1+y}\right)\right)\right) \\
&\quad = \psi\big(\varphi(x), \varphi(x), \varphi(y)\big) \\
&\quad = \psi(2x, 2x, 2y) = \frac{8x}{5},
\end{aligned}$$

and

$$\begin{aligned}
&\phi\big(\max\big\{p(x,y), p(x,fx), p(y,fy)\big\}\big) \\
&\quad = \phi\left(\max\left\{p(x,y), p\left(x, \frac{x^2}{1+x}\right), p\left(y, \frac{y^2}{1+y}\right)\right\}\right) \\
&\quad = \phi\big(\max\{x, x, y\}\big) = \frac{2x}{5(1+x)}.
\end{aligned}$$

Since

$$\frac{2x^2}{1+x} \leq \frac{8x}{5} - \frac{2x}{5(1+x)},$$

we have

$$\begin{aligned}
\varphi\big(p(fx, fy)\big) \leq{}& \psi\big(\varphi\big(p(x,y)\big), \varphi\big(p(x,fx)\big), \varphi\big(p(y,fy)\big)\big) \\
&- \phi\big(\max\big\{p(x,y), p(x,fx), p(y,fy)\big\}\big).
\end{aligned}$$

On the other hand, for $x \in C$ and $y \in A$, without loss of generality, we may assume that $x \leq y$, then it is easy to get the above inequality.

Note that Example 1 satisfies all of the hypotheses of Theorem 3, and we get that 0 is the unique fixed point. □

## 3 Fixed point theorems (II)

In this article, we also recall the notion of a Meir-Keeler function (see [16]). A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$

such that for $t \in [0,\infty)$ with $\eta \le t < \eta+\delta$, we have $\phi(t) < \eta$. We now introduce a new notion of a weaker Meir-Keeler function $\phi : [0,\infty) \to [0,\infty)$ in a partial metric space $(X,p)$ as follows.

**Definition 6** Let $(X,p)$ be a partial metric space. We call $\phi : [0,\infty) \to [0,\infty)$ a weaker Meir-Keeler function in $X$ if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le p(x,y) < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(p(x,y)) < \eta$.

In the section, we denote by $\Phi$ the class of weaker Meir-Keeler functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ in a partial metric space in $(X,p)$ satisfying the following conditions:

($\phi_1$) $\phi(t) > 0$ for $t > 0$, $\phi(0) = 0$;
($\phi_2$) $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
($\phi_3$) for $t_n \in [0,\infty)$,
  (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \phi(t_n) < \gamma$ and
  (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \phi(t_n) = 0$.

And we denote by the class $\Psi$ of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function satisfying $\psi(t) > 0$ for $t > 0$, $\psi(0) = 0$.

First, we state a new notion of cyclic $\mathcal{MK}$-contractions in partial metric spaces as follows.

**Definition 7** Let $(X,p)$ be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \to Y$ is called a cyclic $\mathcal{MK}$-contraction if

(1)
$\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;
(2)
for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$p(fx, fy) \le \phi\big(p(x,y)\big) - \psi\big(p(x,y)\big), \tag{3.1}$$

where
$A_{m+1} = A_1$, $\phi \in \Phi$ and $\psi \in \Psi$.

**Theorem 4** *Let $(X,p)$ be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y \to Y$ be a cyclic $\mathcal{MK}$-contraction. Then $f$ has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.*

*Proof* Given $x_0$ and let $x_{n+1} = fx_n = f^n x_0$, for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \ne x_n$ for any $n = 0, 1, 2, \ldots$. Notice that for any $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by (3.1), we have

$$p(x_n, x_{n+1}) = p(fx_{n-1}, fx_n) \le \phi\big(p(x_{n-1}, x_n)\big) - \psi\big(p(x_{n-1}, x_n)\big).$$

Step 1. We will prove that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \quad \text{that is,} \quad \lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0.$$

Since $f$ is a cyclic $\mathcal{MK}$-contraction, we can conclude that

$$\begin{aligned} p(x_n,x_{n+1}) &\leq \phi\bigl(p(x_{n-1},x_n)\bigr) \\ &\leq \phi(\phi\bigl(p(x_{n-2},x_{n-1})\bigr) = \phi^2\bigl(p(x_{n-2},x_{n-1})\bigr) \\ &\leq \cdots \\ &\leq \phi^n\bigl(p(x_0,x_1)\bigr). \end{aligned}$$

Since $\{\phi^n(p(x_0,x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $0 < \eta$. Then by the definition of a weaker Meir-Keeler function $\phi$, there exists $\delta > 0$ such that for $x_0,x_1 \in X$ with $\eta \leq p(x_0,x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(p(x_0,x_1)) < \eta$. Since $\lim_{n\to\infty}\phi^n(p(x_0,x_1)) = \eta$, there exists $k_0 \in \mathbb{N}$ such that $\eta \leq \phi^k(p(x_0,x_1)) < \delta + \eta$, for all $k \geq k_0$. Thus, we conclude that $\phi^{k_0+n_0}(p(x_0,x_1)) < \eta$. So, we get a contradiction. Therefore, $\lim_{n\to\infty}\phi^n(p(x_0,x_1)) = 0$, and so we have

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{3.2}$$

By $(\mathrm{p}_2)$, we also have

$$\lim_{n\to\infty} p(x_n,x_n) = 0. \tag{3.3}$$

Since $d_p(x,y) \leq 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$, using (3.2) and (3.3), we obtain that

$$\lim_{n\to\infty} d_p(x_n,x_{n+1}) = 0. \tag{3.4}$$

Step 2. We show that $\{x_n\}$ is a Cauchy sequence in the metric space $(Y,d_p)$. We claim that the following result holds.

**Claim** For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r,q \geq n$ with $r - q = 1 \bmod m$, then $d_p(x_r,x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $r_n,q_n \in \mathbb{N}$ with $r_n > q_n \geq n$ with $r_n - q_n = 1 \bmod m$ satisfying

$$d_p(x_{q_n},x_{r_n}) \geq \epsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \geq n$ use, we can choose $r_n$ in such a way it is the smallest integer with $r_n > q_n \geq n$ satisfying $r_n - q_n = 1 \bmod m$ and $d_p(x_{q_n},x_{r_n}) \geq \epsilon$. Therefore, $d_p(x_{q_n},x_{r_n-m}) \leq \epsilon$ and

$$\begin{aligned} \epsilon &\leq d_p(x_{q_n},x_{r_n}) \\ &\leq d_p(x_{q_n},x_{r_n-m}) + \sum_{i=1}^{m} d_p(x_{r_{n-i}},x_{r_{n-i+1}}) \\ &< \epsilon + \sum_{i=1}^{m} d_p(x_{r_{n-i}},x_{r_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_n}, x_{r_n}) = \epsilon. \tag{3.5}$$

On the other hand, we can conclude that

$$\begin{aligned}\epsilon &\le d_p(x_{q_n}, x_{r_n})\\ &\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n})\\ &\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{q_n}) + d_p(x_{q_n}, x_{r_n}) + d_p(x_{r_n}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}).\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{3.6}$$

Since $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and using (3.5) and (3.6), we have that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \frac{\epsilon}{2}, \tag{3.7}$$

and

$$\lim_{n\to\infty} p(x_{q_{n+1}}, x_{r_{n+1}}) = \frac{\epsilon}{2}. \tag{3.8}$$

Since $x_{q_n}$ and $x_{r_n}$ lie in different adjacently labeled sets $A_i$ and $A_{i+1}$ for certain $1 \le i \le m$, by using the fact that $f$ is a cyclic $\mathcal{MK}$-contraction, we have

$$p(x_{q_{n+1}}, x_{r_{n+1}}) = p(fx_{q_n}, fx_{r_n}) \le \phi\bigl(p(x_{q_n}, x_{r_n})\bigr) - \psi\bigl(p(x_{q_n}, x_{r_n})\bigr).$$

Letting $n \to \infty$, by using the condition $\phi_3$ of the function $\phi$, we obtain that

$$\frac{\epsilon}{2} \le \frac{\epsilon}{2} - \psi\left(\frac{\epsilon}{2}\right),$$

and consequently, $\psi(\frac{\epsilon}{2}) = 0$. By the definition of a function $\psi$, we get $\epsilon = 0$ which is a contraction. Therefore, our claim is proved.

In the sequel, we will show that $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$. Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $r, q \ge n_1$ with $r - q = 1 \bmod m$, then

$$d_p(x_r, x_q) \le \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d_p(x_n, x_{n+1}) \le \frac{\varepsilon}{2m}$$

for any $n \ge n_2$.

Let $r, q \geq \max\{n_1, n_2\}$ and $r > q$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $r - q = k \bmod m$. Therefore, $r - q + j = 1 \bmod m$ for $j = m - k + 1$, and so we have

$$\begin{aligned} d_p(x_q, x_r) &\leq d_p(x_q, x_{r+j}) + d_p(x_{r+j}, x_{r+j-1}) + \cdots + d_p(x_{r-1}, x_r) \\ &\leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \\ &\leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$.

Step 3. We show that $f$ has a fixed point $\nu$ in $\bigcap_{i=1}^m A_i$.

Since $Y$ is closed, the subspace $(Y, p)$ is complete. Then from Lemma 1, we have that $(Y, d_p)$ is complete. Thus, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, \nu) = 0.$$

And it follows from Lemma 1 that we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.9}$$

On the other hand, since the sequence $\{x_n\}$ is a Cauchy sequence in the metric space $(Y, d_p)$, we also have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0.$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, we can deduce that

$$\lim_{n\to\infty} p(x_n, x_m) = 0. \tag{3.10}$$

Since $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of $X$ with respect to $f$, the sequence $\{x_n\}$ has infinite terms in each $A_i$ for $i \in \{1, 2, \dots, m\}$. Now, for all $i = 1, 2, \dots, m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to $\nu$. Using (3.9) and (3.10), we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n\to\infty} p(x_{n_k}, \nu) = 0.$$

By (3.1),

$$\begin{aligned} p(x_{n_{k+1}}, f\nu) &= p(fx_{n_k}, f\nu) \\ &\leq \phi\bigl(p(x_{n_k}, \nu)\bigr) - \psi\bigl(p(x_{n_k}, \nu)\bigr) \\ &\leq \phi\bigl(p(x_{n_k}, \nu)\bigr). \end{aligned}$$

Letting $k \to \infty$, we have

$$p(\nu, f\nu) \leq 0,$$

and so $\nu = f\nu$.

Step 4. Finally, to prove the uniqueness of the fixed point, let $\mu$ be another fixed point of $f$ in $\bigcap_{i=1}^m A_i$. By the cyclic character of $f$, we have $\mu, \nu \in \bigcap_{i=1}^n A_i$. Since $f$ is a cyclic weaker $\mathcal{MK}$-contraction, we have

$$\begin{aligned} p(\nu,\mu) &= p(\nu, f\mu) \\ &= \lim_{n\to\infty} p(x_{n_{k+1}}, f\mu) \\ &= \lim_{n\to\infty} p(fx_{n_k}, f\mu) \\ &\le \lim_{n\to\infty} \bigl[\phi\bigl(p(x_{n_k},\mu)\bigr) - \psi\bigl(p(x_{n_k},\mu)\bigr)\bigr] \\ &\le p(\nu,\mu) - \psi\bigl(p(\nu,\mu)\bigr), \end{aligned}$$

and we can conclude that

$$\psi\bigl(p(\nu,\mu)\bigr) = 0,$$

which implies $p(\nu,\mu) = 0$. So, we have $\mu = \nu$. We complete the proof. □

The following provides an example for Theorem 4.

**Example 2** Let $X = [0,1]$ and $A = [0,1]$, $B = [0,\frac{1}{2}]$, $C = [0,\frac{1}{4}]$. We define the partial metric $p$ on $X$ by

$$p(x,y) = \max\{x,y\} \quad \text{for all } x, y \in X,$$

and define the function $f : X \to X$ by

$$f(x) = \frac{x^2}{1+x} \quad \text{for all } x \in X.$$

Now, we let $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ be

$$\phi(t) = \frac{4t}{5} \quad \text{and} \quad \psi(t) = \frac{t}{5(1+t)}.$$

Then $f$ is a cyclic $\mathcal{MK}$-contraction and 0 is the unique fixed point.

By Theorem 4, it is easy to get the following corollary.

**Corollary 1** *Let $(X,p)$ be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of $X$, $Y = \bigcup_{i=1}^m A_i$ and let $f : Y \to Y$. Assume that*

(1) *$\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*

(2) *for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,*

$$p(fx, fy) \le \phi\bigl(p(x,y)\bigr),$$

*where $A_{m+1} = A_1$ and $\phi \in \Phi$.*

*Then $f$ has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.*

**Competing interests**
The author declares that they have no competing interests.

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